comillas.edu

COMILLAS
UNIVERSIDAD PONTIFICIA
ICAI

Time Series Forecasting

Chapter 2: ARMA models

September 2024

Contents

comillas.edu

1

Basic Linear Processes

Basic Linear Processes
Definition

- A linear process can be represented as a linear combination of random variables (Box-Jenkins):

$$y[t] = \mu + \sum_{i=0}^{\infty} \psi_i \, \varepsilon[t-i]$$

where μ is the mean of $y[t]$, $\psi_0 = 1$ and $\{\varepsilon[t]\}$ is a sequence of iid random variables with zero mean and well defined distribution.

- We will focus on 3 types of linear processes:
  - White noise processes
  - Autoregressive processes
  - Moving average processes

White noise process

- **Definition**: sequence of uncorrelated random variables, identically distributed with zero mean and constant variance.

- **General expression:** $y[t] = \varepsilon[t]$

- **Properties:**

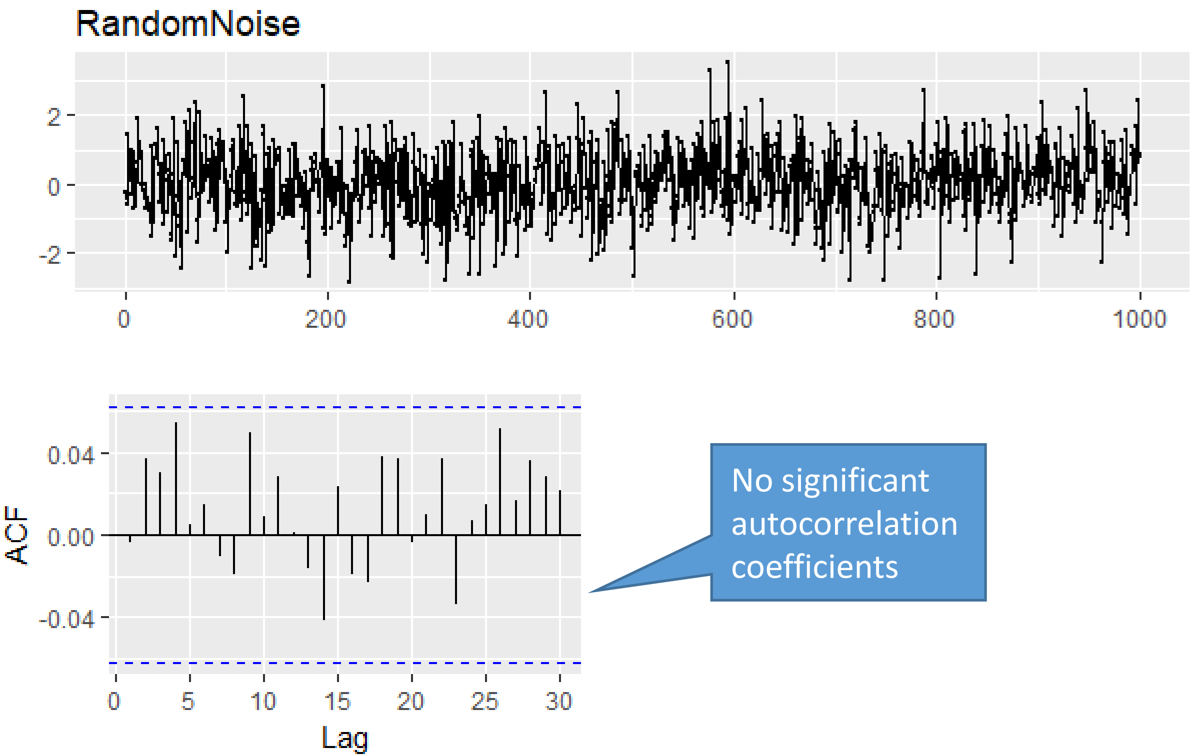
$$E(\varepsilon[t]) = 0 \qquad \forall t$$

$$E(\varepsilon[t]^2) = \sigma^2 \qquad \forall t$$

$$E(\varepsilon[t]\varepsilon[t']) = 0 \qquad \forall t \neq t'$$

- **Others:**
  - Independent or strict white noise
  - Gaussian white noise

Basic Linear Processes
White noise process



No significant autocorrelation coefficients

Basic Linear Processes
Autoregressive processes

- Process *AR(p)* (Yule, 1927):

$$y[t] = \cancel{\delta} + \varphi_1 y[t-1] + \varphi_2 y[t-2] + \ldots + \varphi_p y[t-p] + \varepsilon[t]$$

- Using the backshift operator B: $By[t] = y[t\text{-}1]$

$$\boxed{\varphi(B)\, y[t] = \varepsilon[t]}$$

where:

$$\varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p$$

Autoregressive processes

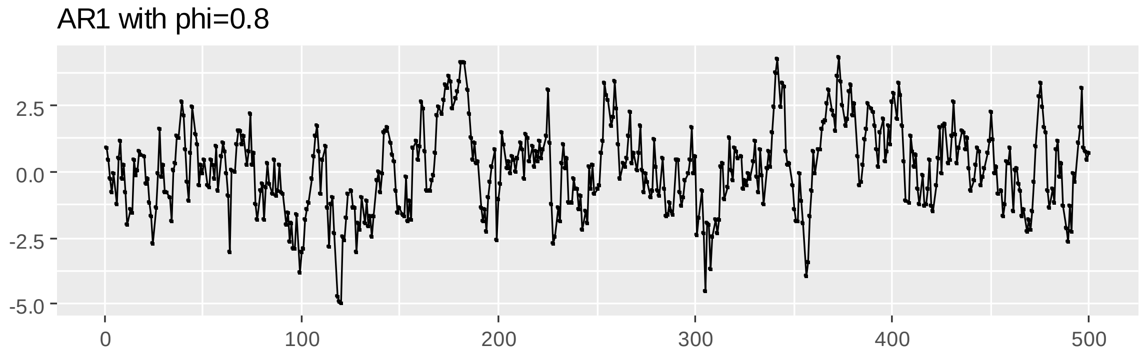
- For an *AR(p)* to be stationary, the roots of its characteristic polynomial:

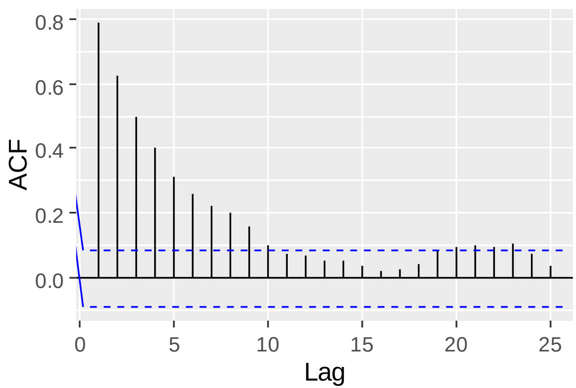
$$1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p = 0$$

have to lie outside the unit circle.

Basic Linear Processes

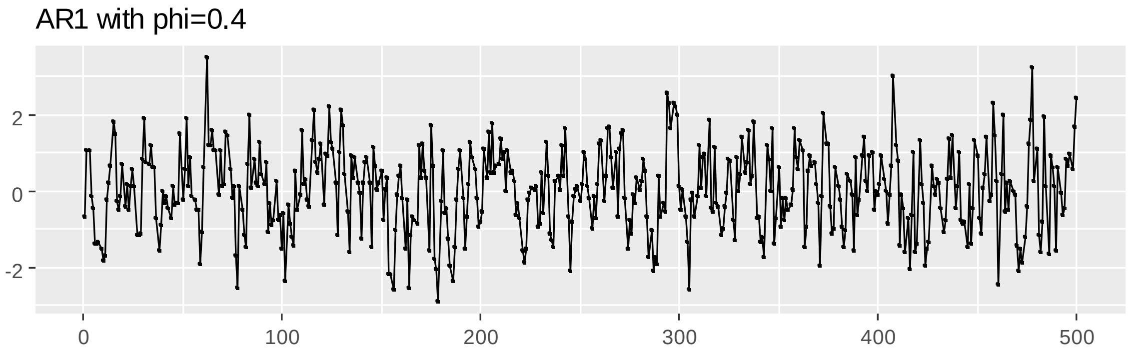
Autoregressive processes

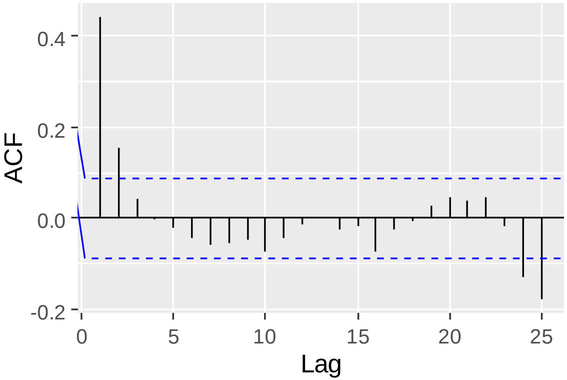




Exponential decay

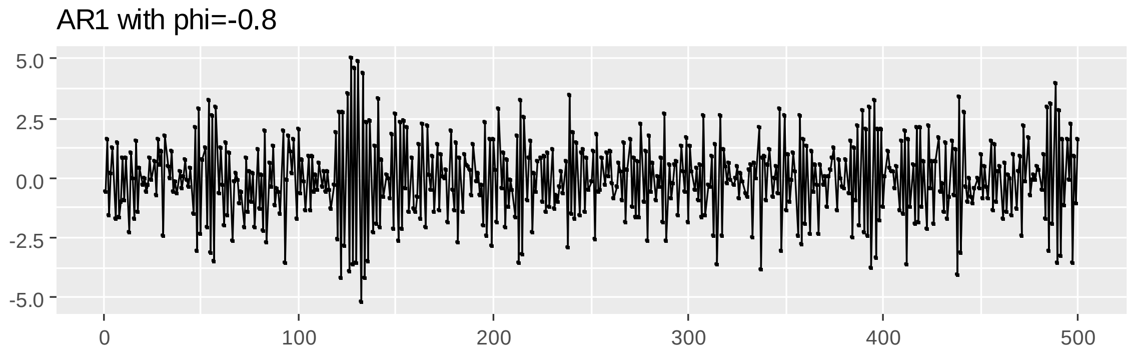
Basic Linear Processes
Autoregressive processes

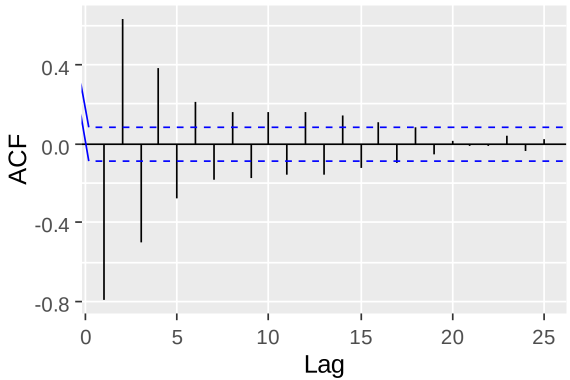




Exponential decay

Basic Linear Processes
Autoregressive processes

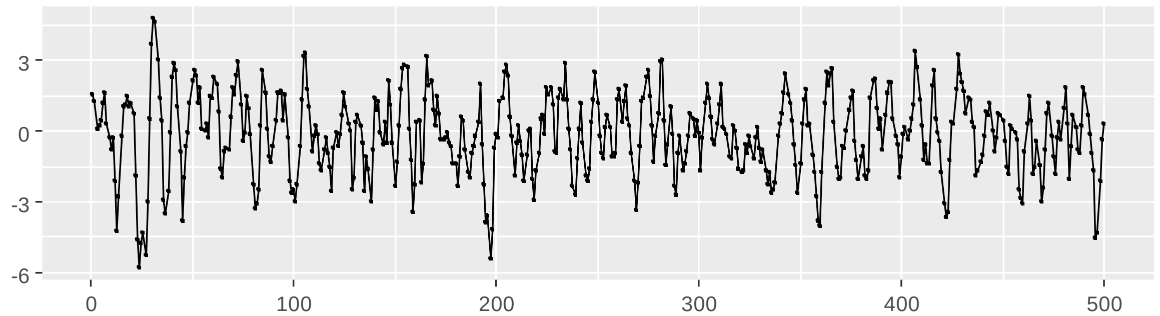


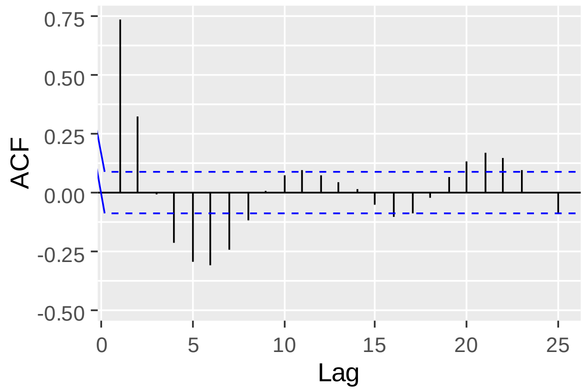


Exponential decay

Basic Linear Processes
Autoregressive processes

AR2 with phi1=1.1 and phi2=-0.5

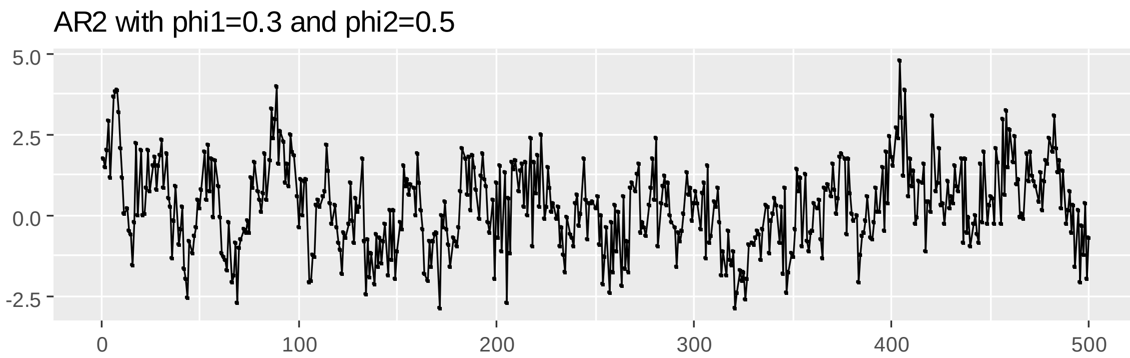


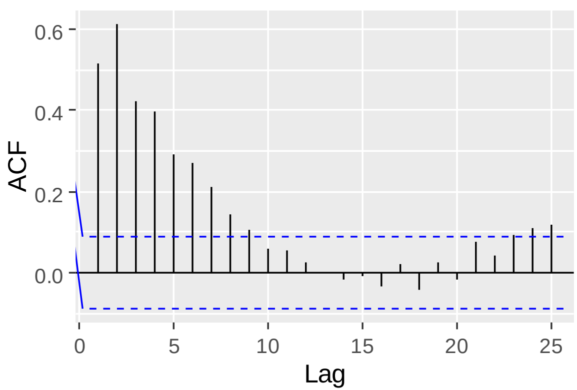


Sinusoidal decay

comillas.edu

Basic Linear Processes
Autoregressive processes





Double exponential decay

Basic Linear Processes

Moving Average processes

- *MA(q)* process (Yule, 1921):

$$y[t] = \cancel{\delta} + \varepsilon[t] - \theta_1 \varepsilon[t-1] - \theta_2 \varepsilon[t-2] - \ldots - \theta_q \varepsilon[t-q]$$

or: $$\boxed{y[t] = \theta(B)\varepsilon[t]}$$

where:

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q$$

Basic Linear Processes

Moving Average processes

- It is possible to write any stationary *AR(p)* model as an *MA(∞)*:

$$
\begin{aligned}
y[t] &= \varphi y[t-1] + \varepsilon[t] \\
&= \varphi(\varphi y[t-2] + \varepsilon[t-1]) + \varepsilon[t] \\
&= \varphi^2 y[t-2] + \varphi\varepsilon[t-1] + \varepsilon[t] \\
&= \varphi^3 y[t-3] + \varphi^2\varepsilon[t-2] + \varphi\varepsilon[t-1] + \varepsilon[t]
\end{aligned}
$$

- For a *MA(q)* process to be invertible, the roots of the polynomial:

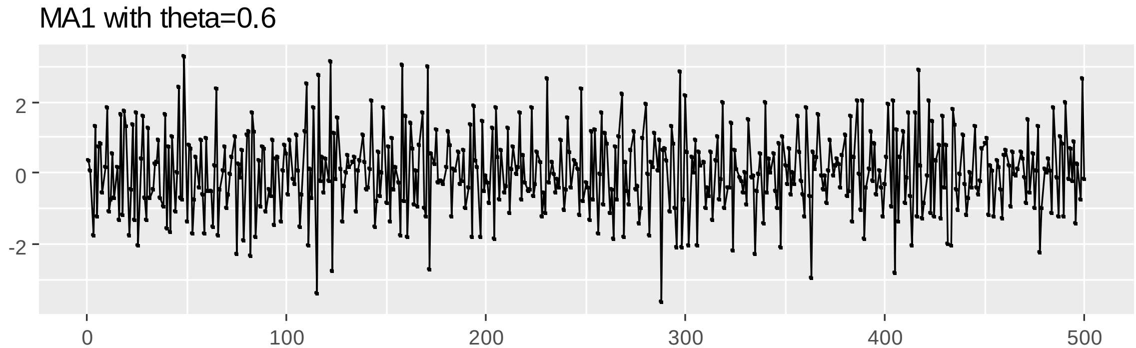
$$1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q = 0$$

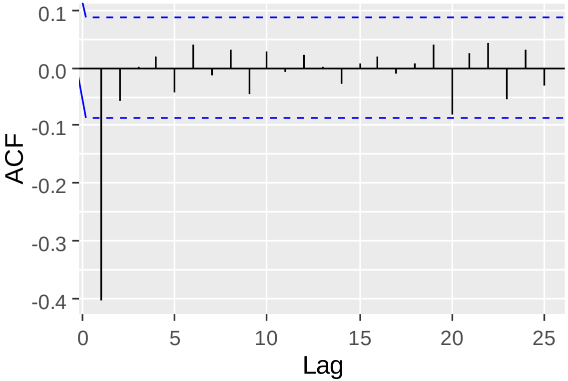
have to lie outside the unit circle.

If the *MA(q)* process is invertible, then it can be written as an *AR(∞)*.

Basic Linear Processes

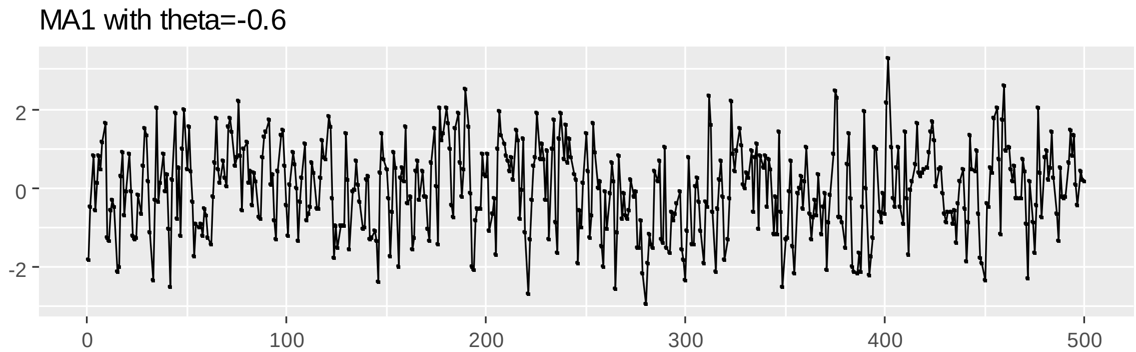
Moving Average processes

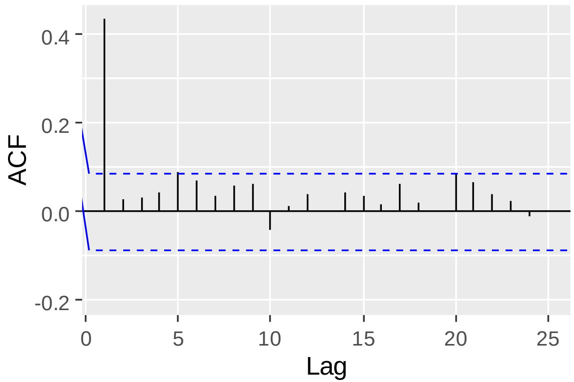




Only one significant autocorrelation coefficient

Basic Linear Processes
Moving Average processes

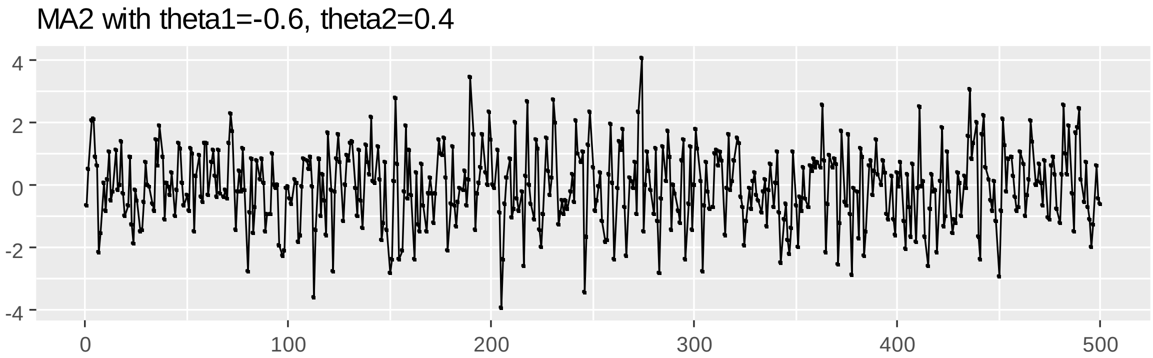


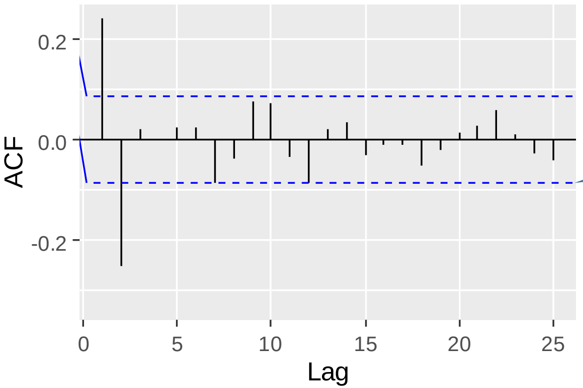


Only one significant autocorrelation coefficient

Basic Linear Processes

Moving Average processes





Only two significant autocorrelation coefficients

ARMA processes

- *ARMA(p,q)* process (Wold, 1938):

$$y[t]-\varphi_1 y[t-1]-\ldots-\varphi_p y[t-p]=\cancel{\delta}+\varepsilon[t]-\theta_1\varepsilon[t-1]-\ldots-\theta_q\varepsilon[t-q]$$

or: $\boxed{\varphi(B)y[t]=\theta(B)\varepsilon[t]}$

where:

$$\varphi(B)=1-\varphi_1 B-\varphi_2 B^2-\ldots-\varphi_p B^p$$

$$\theta(B)=1-\theta_1 B-\theta_2 B^2-\ldots-\theta_q B^q$$

ARMA processes

- For an *ARMA(p,q)* process to be stationary, the roots of the polynomial:

$$1 - \varphi_1 B - \varphi_2 B^2 - \ldots - \varphi_p B^p = 0$$

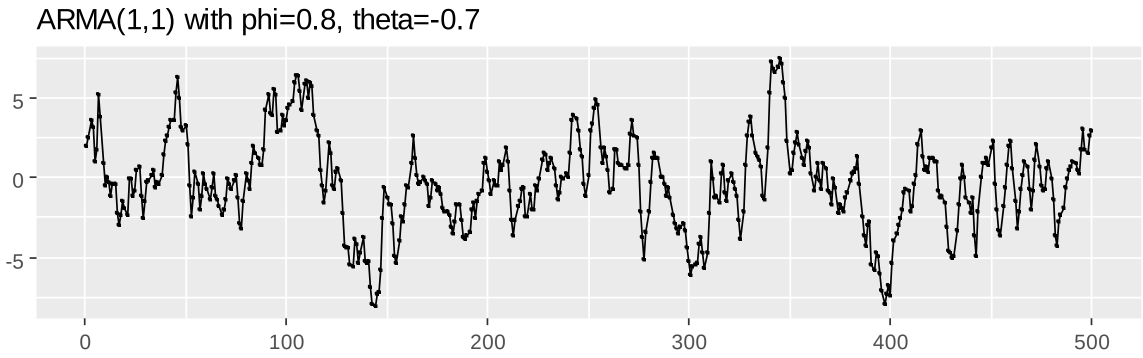
have to lie outside the unit circle.

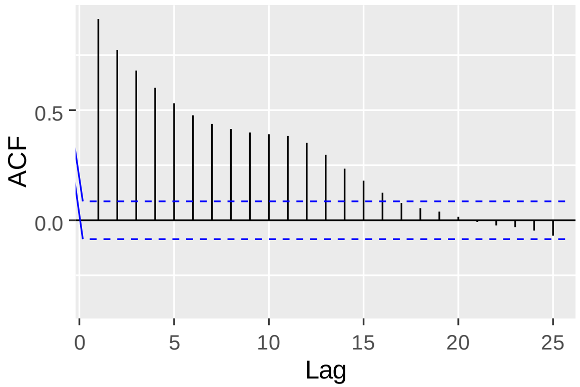
- For an *ARMA(p,q)* process to be invertible, the roots of the polynomial:

$$1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q = 0$$

have to lie outside the unit circle.

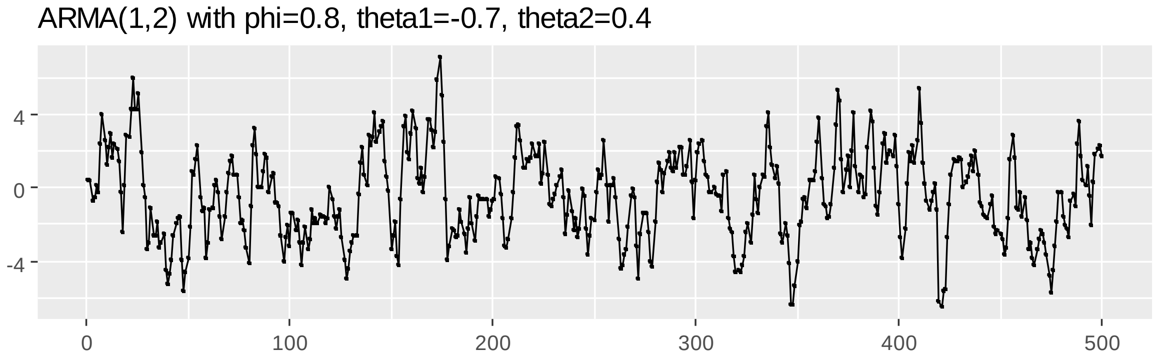
Basic Linear Processes
ARMA processes

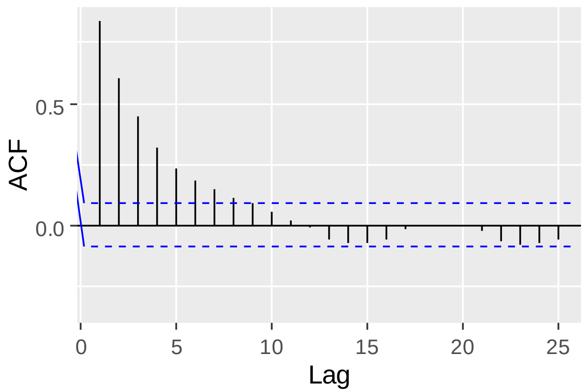




Complex ACF pattern

Basic Linear Processes
ARMA processes





Complex ACF pattern

comillas.edu

comillas.edu

2

ARMA Model Identification

ARMA Model Identification
Sample Autocorrelation Function (ACF)

- Autocorrelation:

$$\rho_k = \frac{\gamma_k}{\gamma_0} \quad \rightarrow \quad \hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0} = \frac{\sum_{t=1}^{N-k} (y[t+k]-\bar{y})(y[t]-\bar{y})}{\sum_{t=1}^{N} (y[t]-\bar{y})^2}$$

- Under the assumption $\rho_k=0$,

$$\hat{\rho}_k \sim N\left(0, \sigma^2_{\hat{\rho}_k}\right)$$

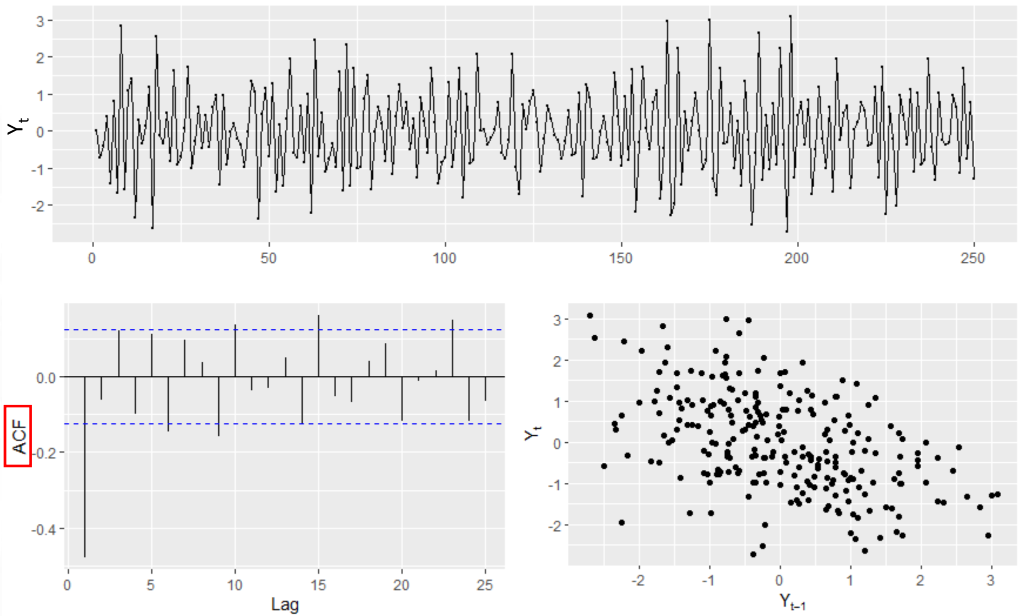
if, in addition, *{y[t]}* is a *MA(q)* process, then: $\sigma^2_{\hat{\rho}_k} \approx \frac{1}{N}(1+2\sum_{i=1}^{k-1}\hat{\rho}_i^2)$

and we can stablish the 95% confidence interval:

$$\boxed{\hat{\rho}_k \in \left[-1.96\sqrt{\sigma^2_{\hat{\rho}_k}}\ ;\ +1.96\sqrt{\sigma^2_{\hat{\rho}_k}}\right]}$$

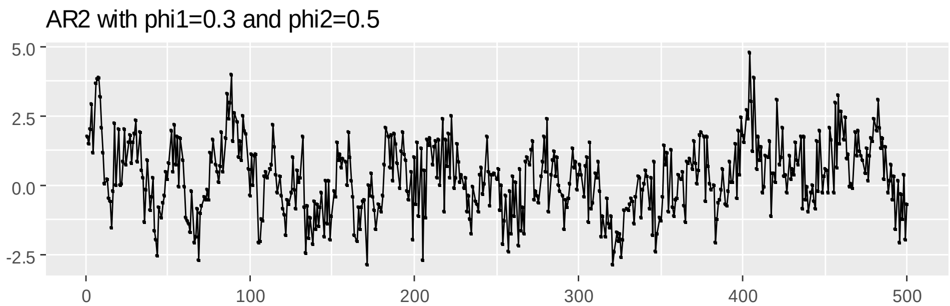
Fundamental concepts
Stationary Processes

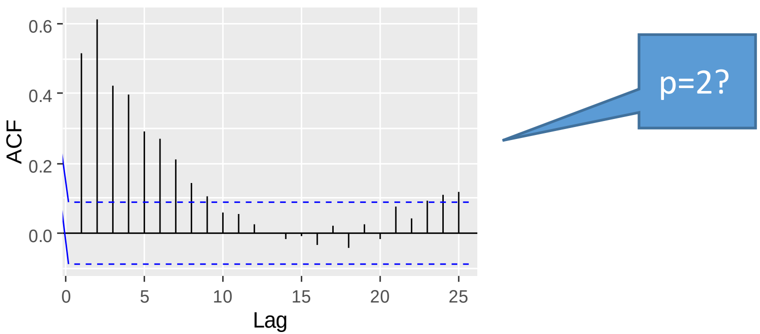
- Correlogram $=\{\hat{\rho}_k\}$ for $k=1,\ldots$ (not recommended for $k>N/4$)



ARMA Model Identification
Partial Autocorrelation Function (PACF)

- For an *AR(p) process*, the ACF decays after *k=p*, but it never reaches 0 $\Rightarrow$ it is not easy to identify an AR process from its ACF





ARMA Model Identification
Partial Autocorrelation Function (PACF)

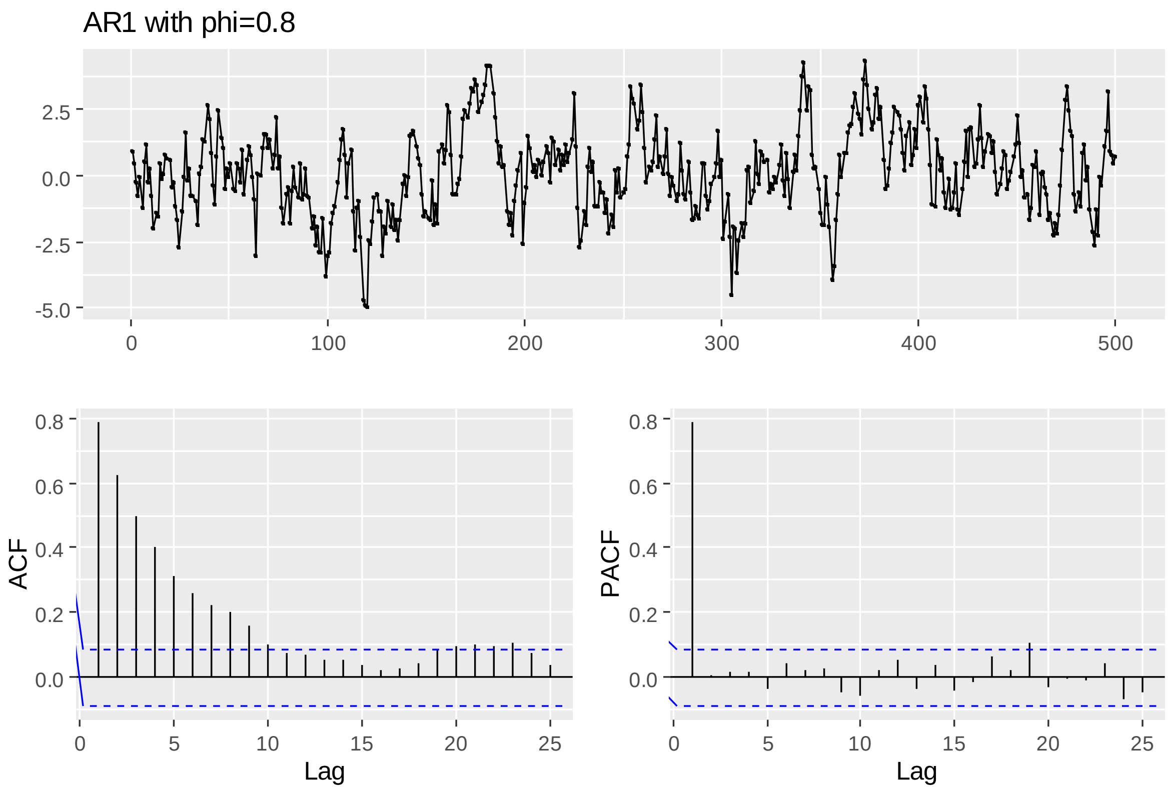
- The PACF can be obtained by linear regression, interpreting each coefficient ϕ_{kk} as the partial correlation between *y[t]* and *y[t-k]* after having eliminated in both variables the effects of *y[t-1],..,y[t-k+1]:*

$$\hat{y}[t] = \hat{\phi}_{1,1} y[t-1]$$
$$\hat{y}[t] = \hat{\phi}_{2,1} y[t-1] + \hat{\phi}_{2,2} y[t-2]$$
$$\ldots$$

- For an *AR(p)* process and *k>p:*

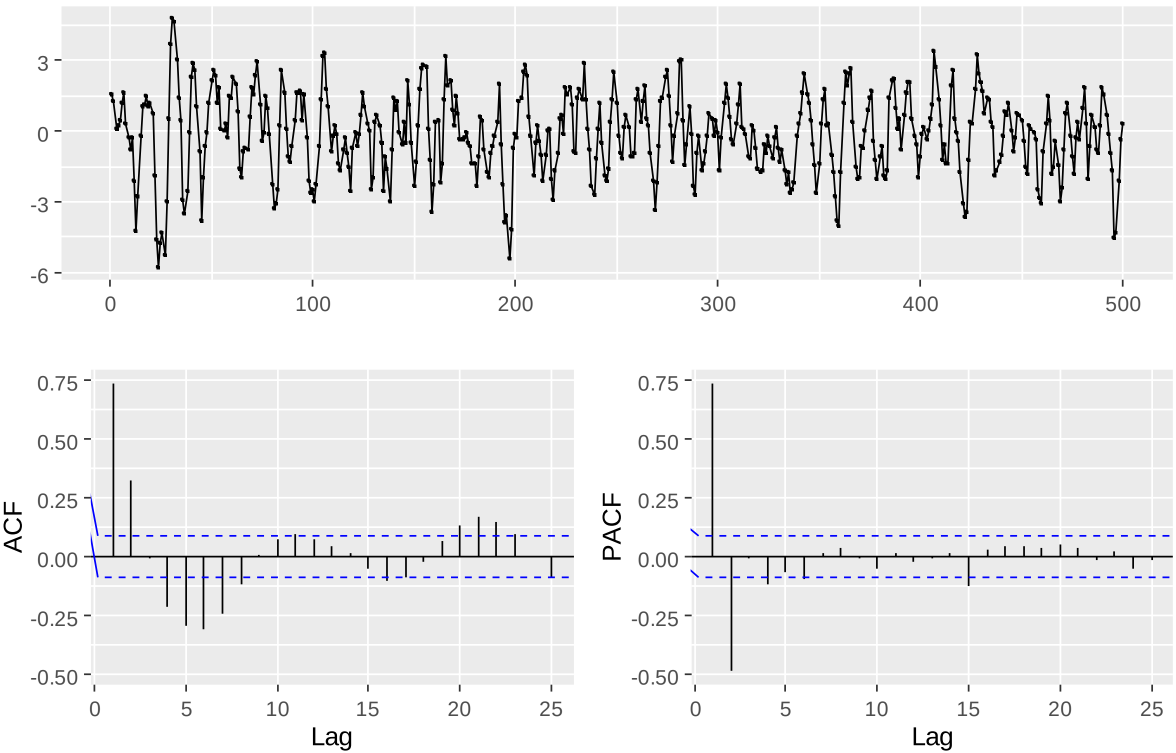
$$\boxed{\hat{\phi}_{kk} \sim N\left(0\ ,\ \frac{1}{N}\right) \quad \Rightarrow \quad \hat{\phi}_{kk} \in \left[-\frac{1.96}{\sqrt{N}}, \frac{1.96}{\sqrt{N}}\right]}$$

ARMA Model Identification
ACF/PACF Examples

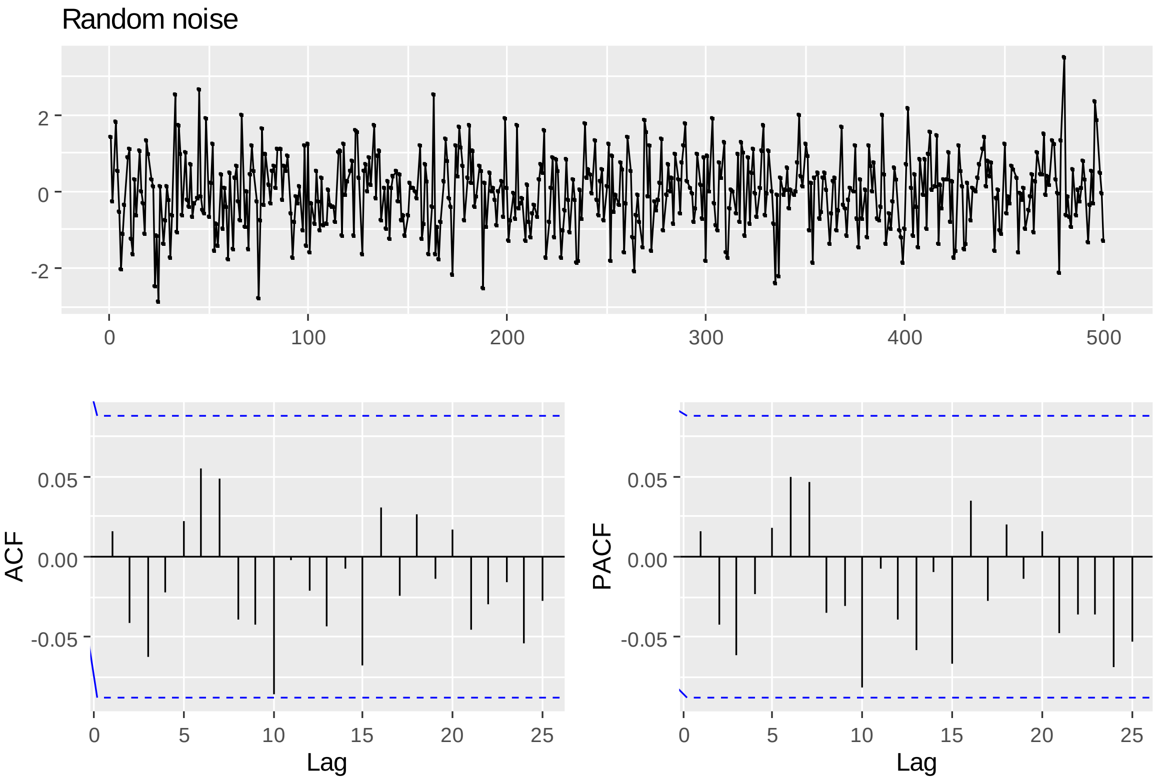


ARMA Model Identification
ACF/PACF Examples

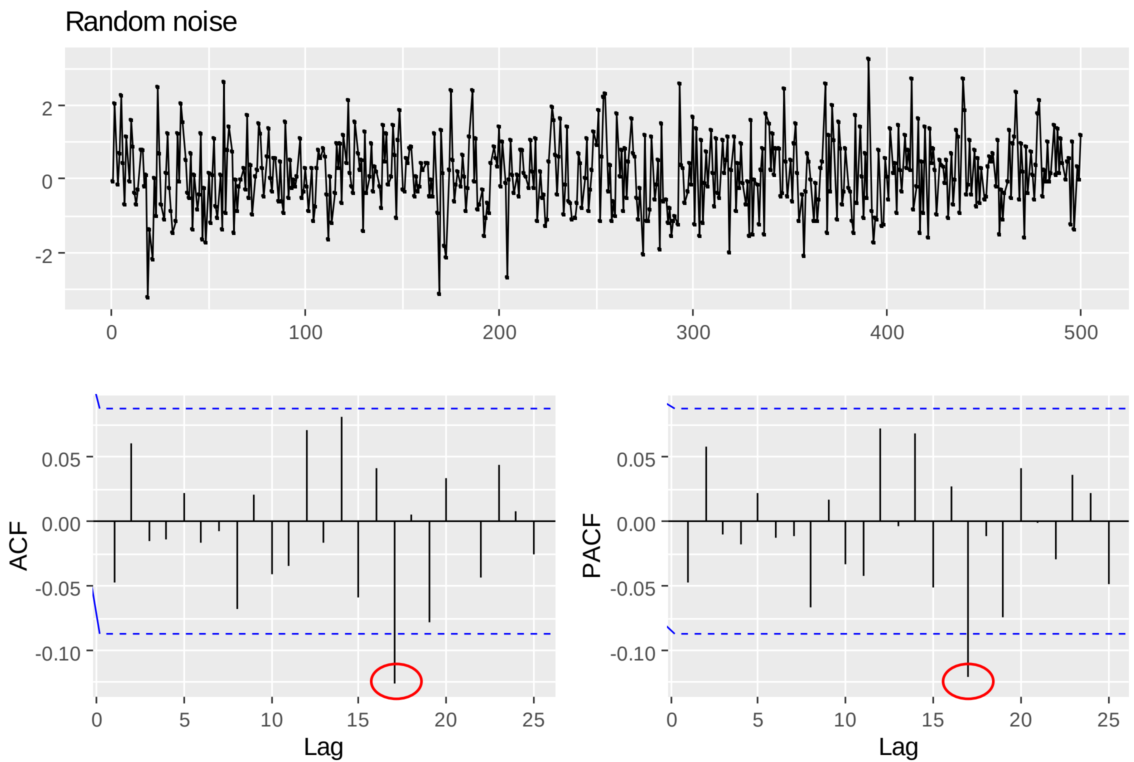
AR2 with phi1=1.1 and phi2=-0.5



ARMA Model Identification
ACF/PACF Examples

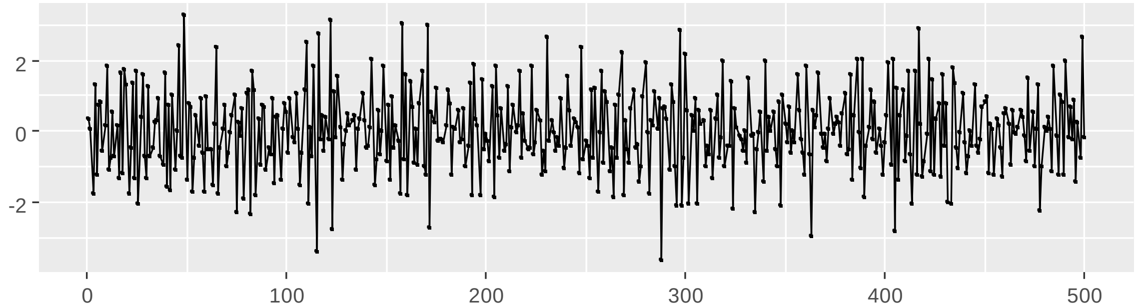


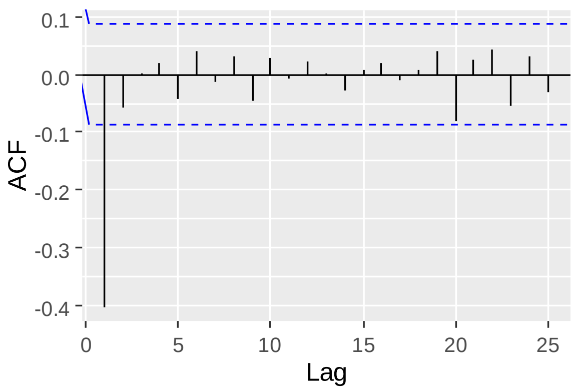
ARMA Model Identification
ACF/PACF Examples

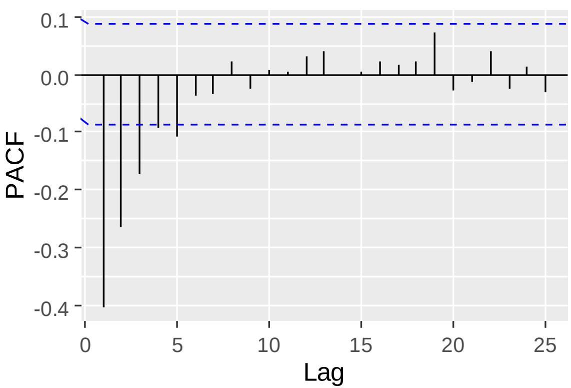


ARMA Model Identification
ACF/PACF Examples

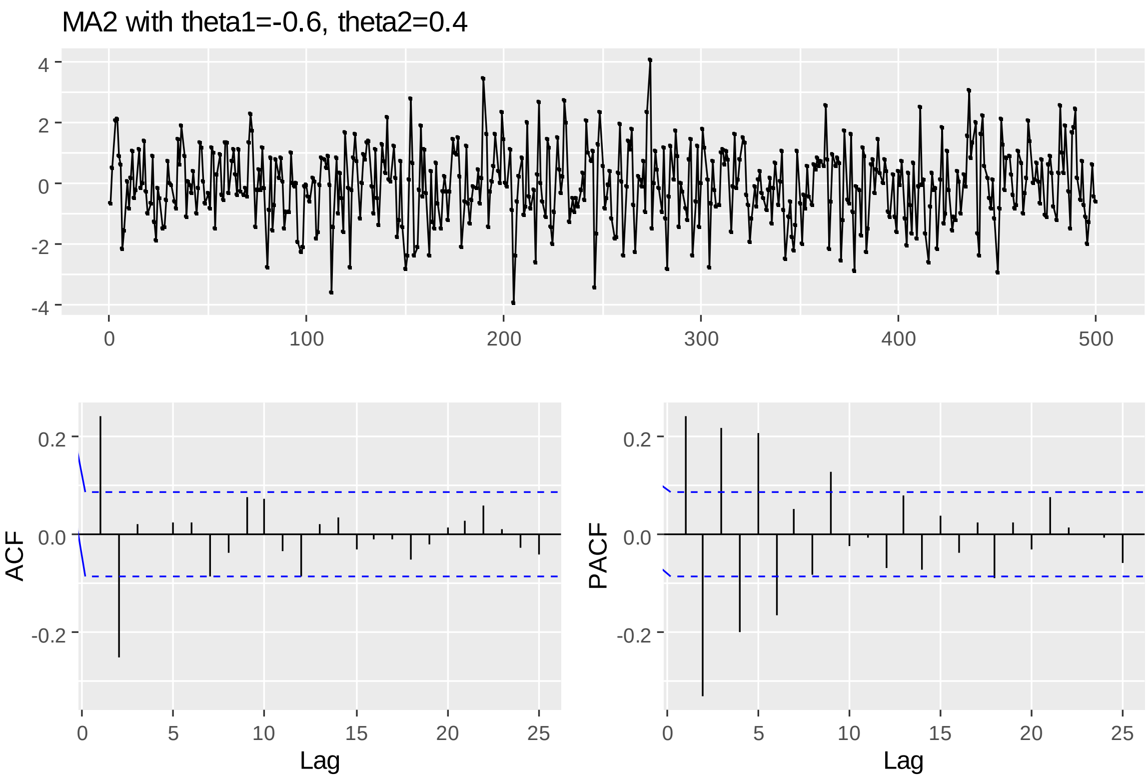
MA1 with theta=0.6







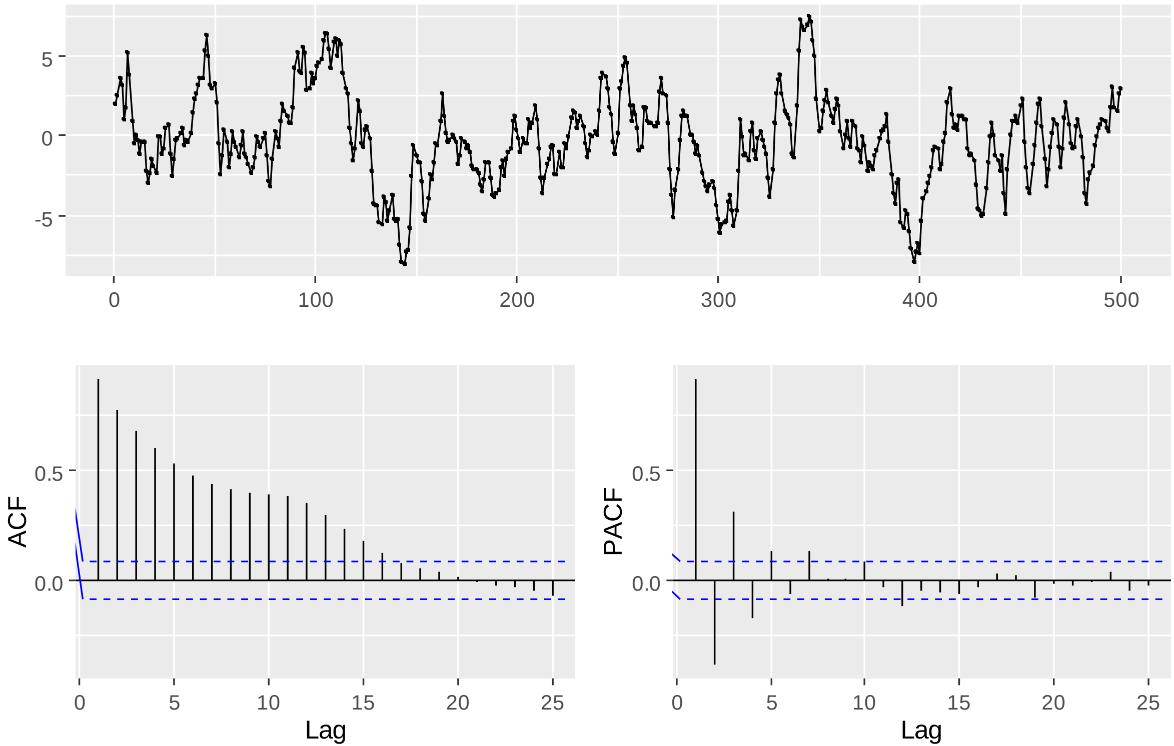
ARMA Model Identification
ACF/PACF Examples



Basic Linear Processes

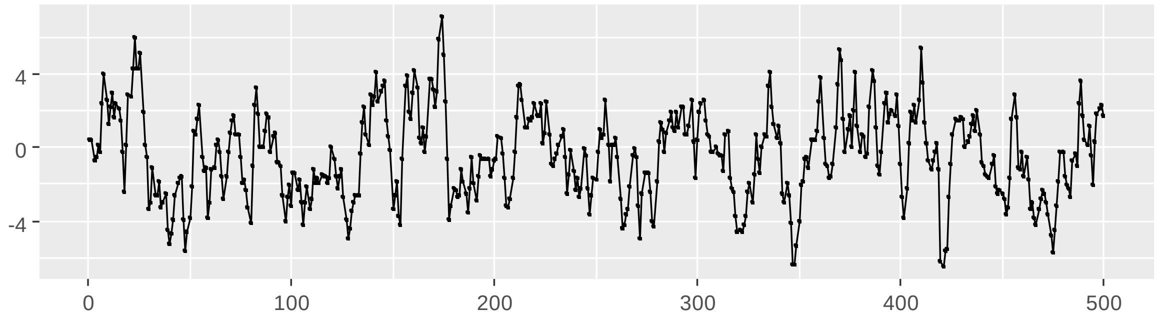
ARMA processes

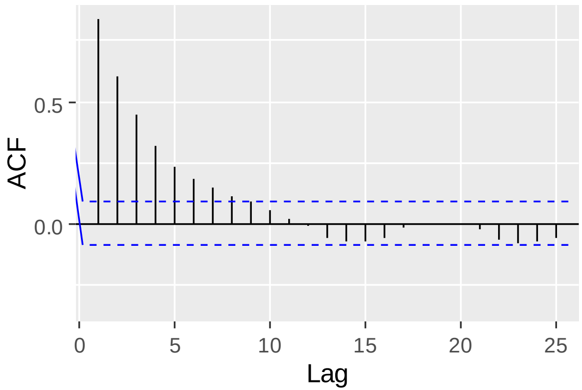
ARMA(1,1) with phi=0.8, theta=-0.7

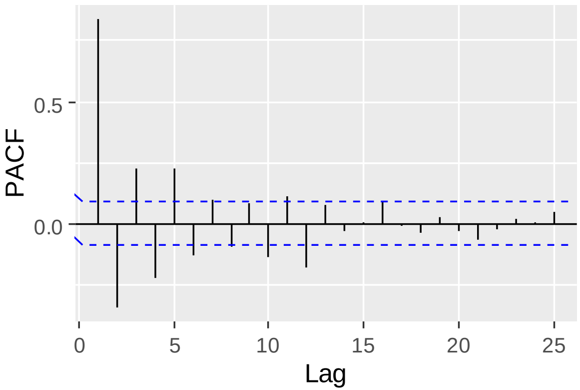


Basic Linear Processes
ARMA processes

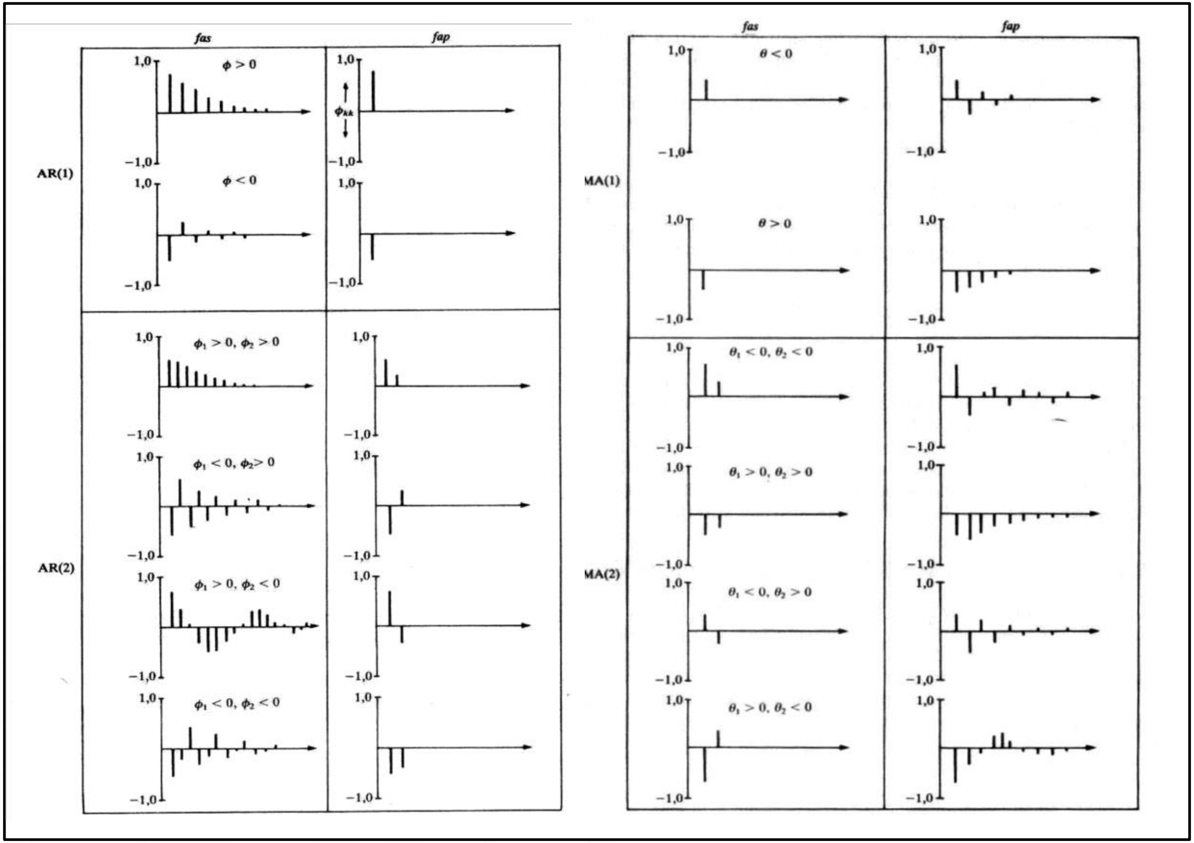
ARMA(1,2) with phi=0.8, theta1=-0.7, theta2=0.4



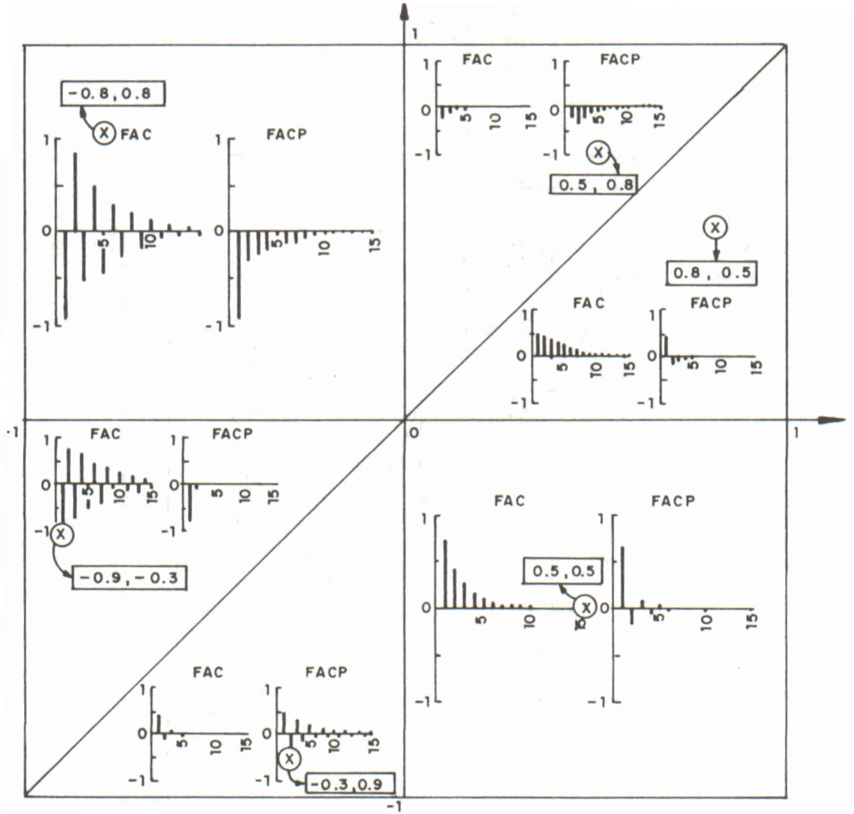




ARMA Model Identification
Identification of AR and MA processes



ARMA Model Identification
Identification of ARMA(1,1) processes



3

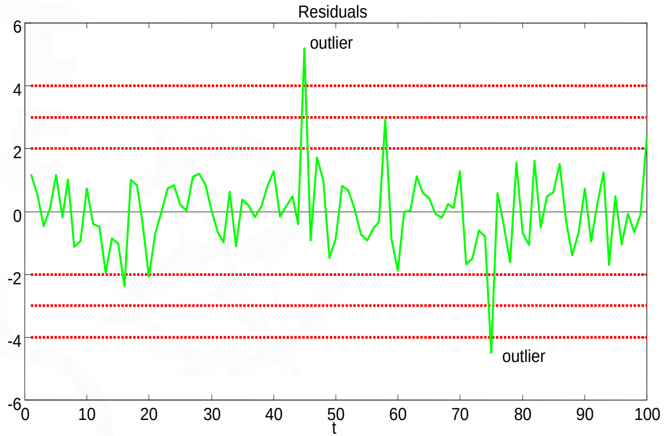
ARMA Model Diagnosis

ARMA Model Diagnosis
The ideal model

- Residuals = White Noise (Gausian)
- Stationary and invertible
- Coefficients are statistically significant and uncorrelated
- Model coefficients are sufficient to represent the series
- High degree of fit compared with other models

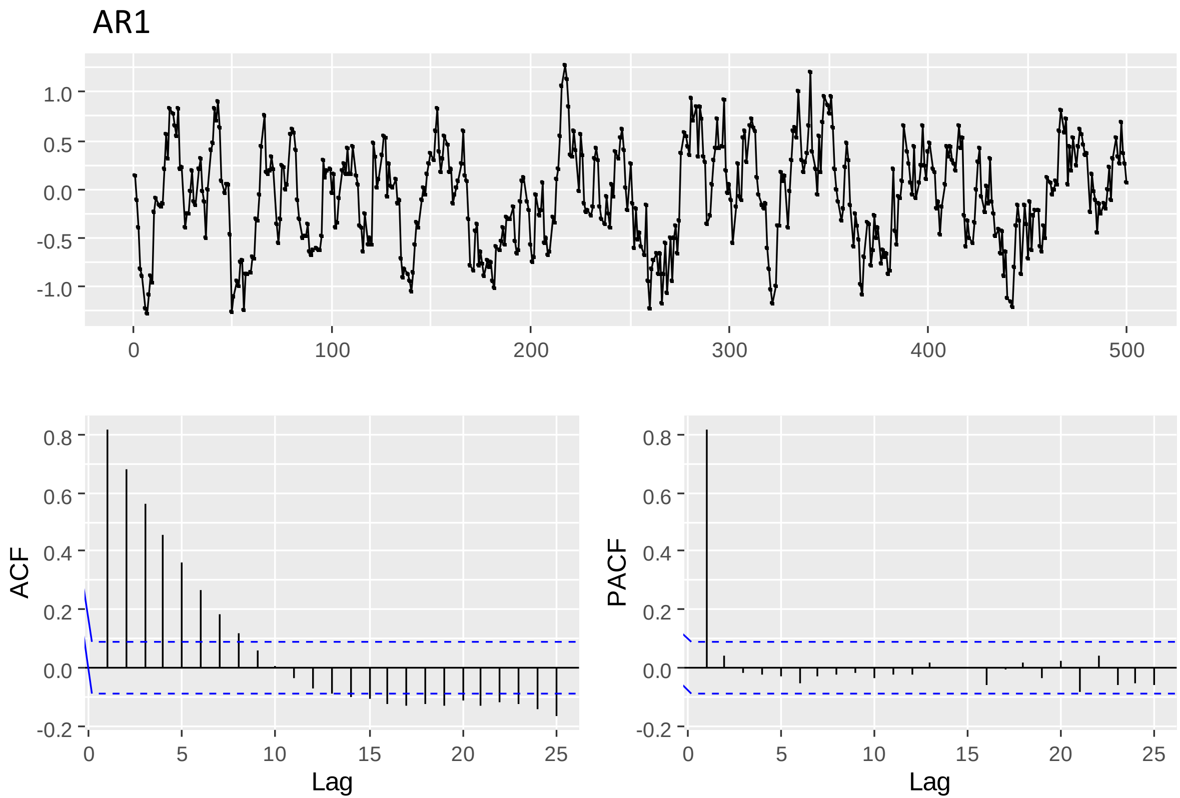
ARMA Model Diagnosis
Residual analysis

- **Plot of the standarized residuals** with different confidence limits ($\pm 2\sigma_\varepsilon$, $\pm 3\sigma_\varepsilon$, $\pm 4\sigma_\varepsilon$)

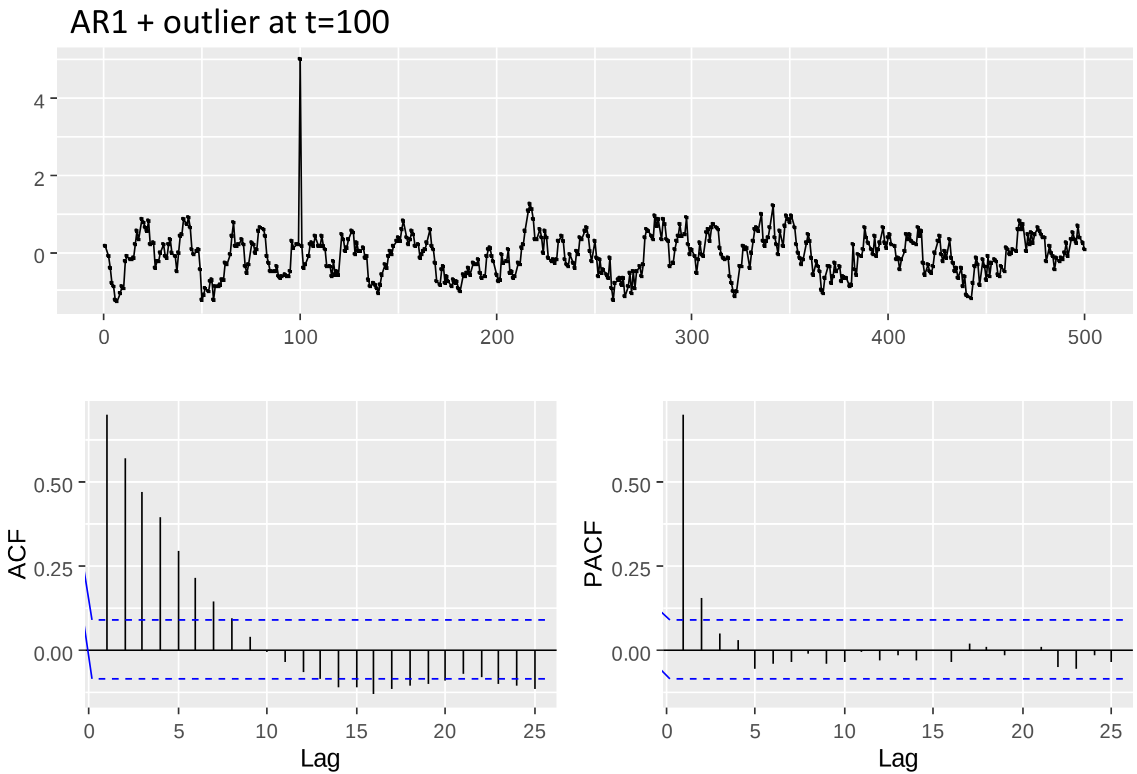


- Check for heteroskedasticity (constant variance)
- Detection of outliers

ARMA Model Diagnosis
Residual analysis: effect of an outlier



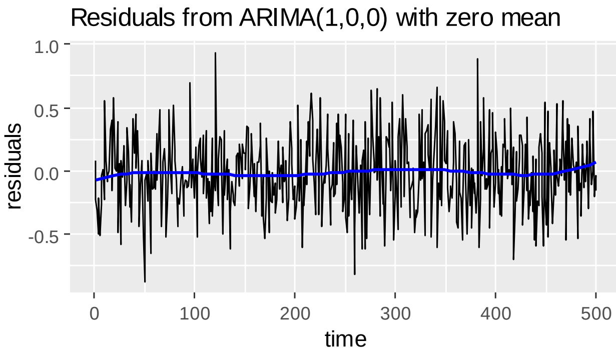
ARMA Model Diagnosis
Residual analysis: effect of an outlier

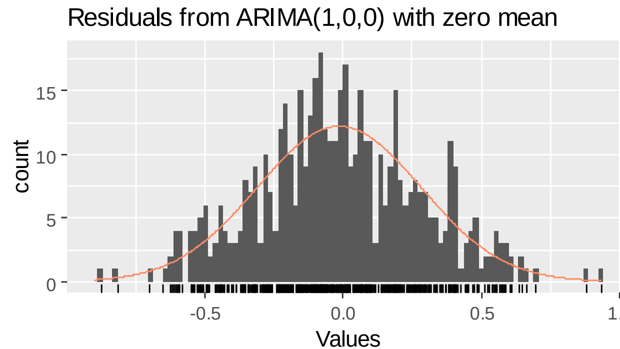


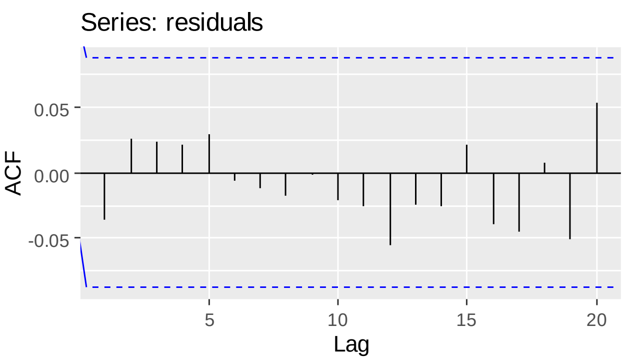
ARMA Model Diagnosis
Residual analysis

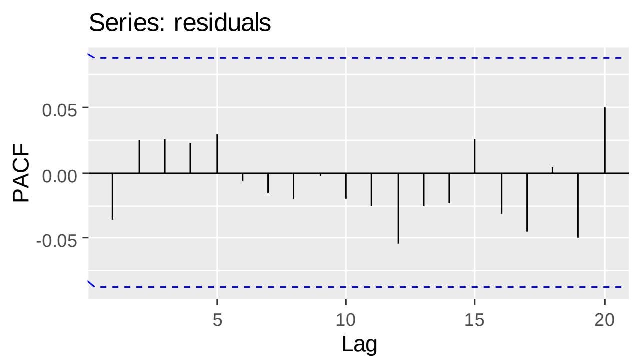
```
     Estimate Std.Error z-value Pr(>|z|)
ar1 0.799547 0.026679 29.969 < 2.2e-16 ***
```

AR1 with phi=0.8





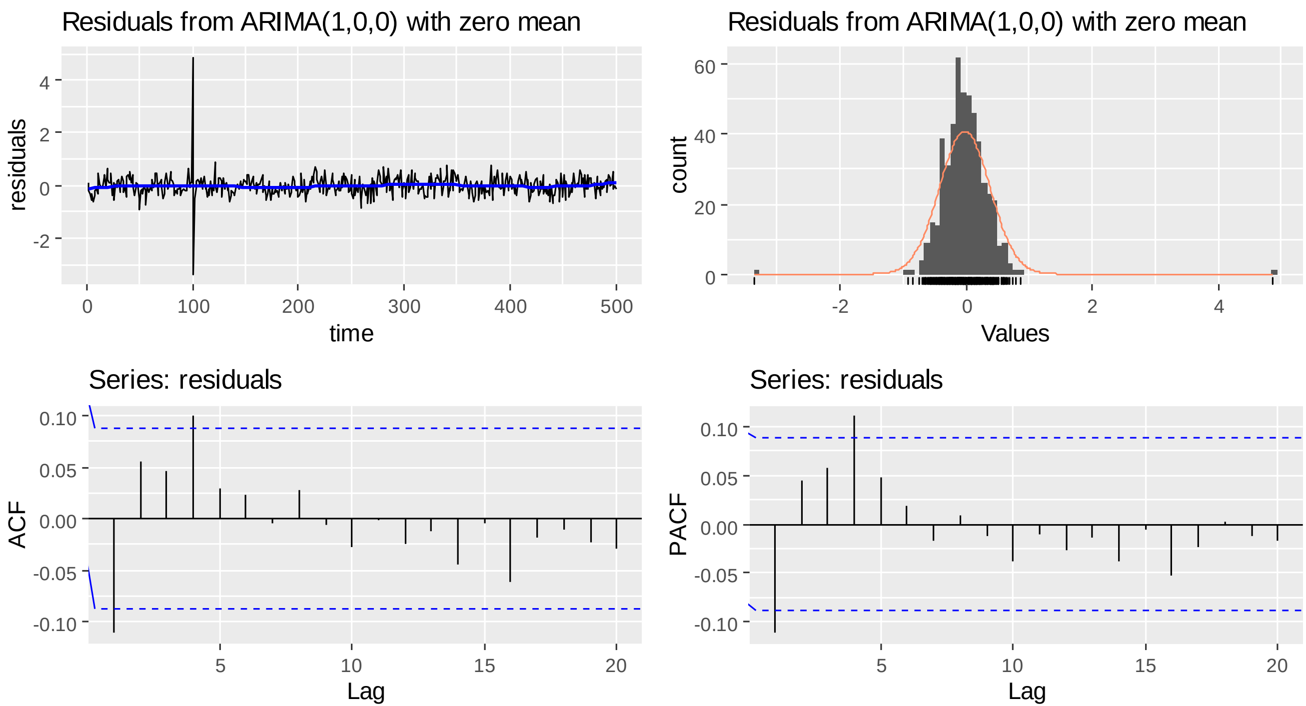




ARMA Model Diagnosis
Residual analysis

```
     Estimate Std.Error z-value Pr(>|z|)
ar1 0.664459 0.033299 19.954 < 2.2e-16 ***
```

AR1 with phi=0.8 + outlier at t=100



ARMA Model Diagnosis
Residual analysis

- Check the degree of significance of each autocorrelation coefficient. For a white noise process *{$\varepsilon[t]$}*:

$$r_\tau \ \& \ \phi_{\tau\tau} \sim N(0\,,\,1/N) \qquad \text{for all } \tau$$

([*Anderson, 1942], [Barlett, 1946], [Quenouille, 1949]*).

Therefore we can then stablish the 95% confidence interval:

$$|r_\tau| < \frac{1.96}{\sqrt{N}}$$

$$|\varphi_{\tau\tau}| < \frac{1.96}{\sqrt{N}}$$

ARMA Model Diagnosis
Residual analysis

- *Portmanteau* test of the residuals:

  Ljung & Box statistic (1978)

$$Q = N(N+2)\sum_{\tau=1}^{M}\frac{r_\tau^2}{N-\tau}$$

Under the null hypothesis that the residuals are independent, Q is distributed according to a χ^2 with *M-p-q* degrees of freedom.

$$\boxed{\text{If } Q < \chi^2_{M-p-q}(\alpha) \text{ accept } H_0}$$

If $Q > \chi^2_{M-p-q}(\alpha)$ reject H_0 (tipical values of α :*5%* or *1%*)

ARMA Model Diagnosis
Residual analysis

E: Critical values for χ^2 statistic

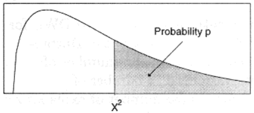
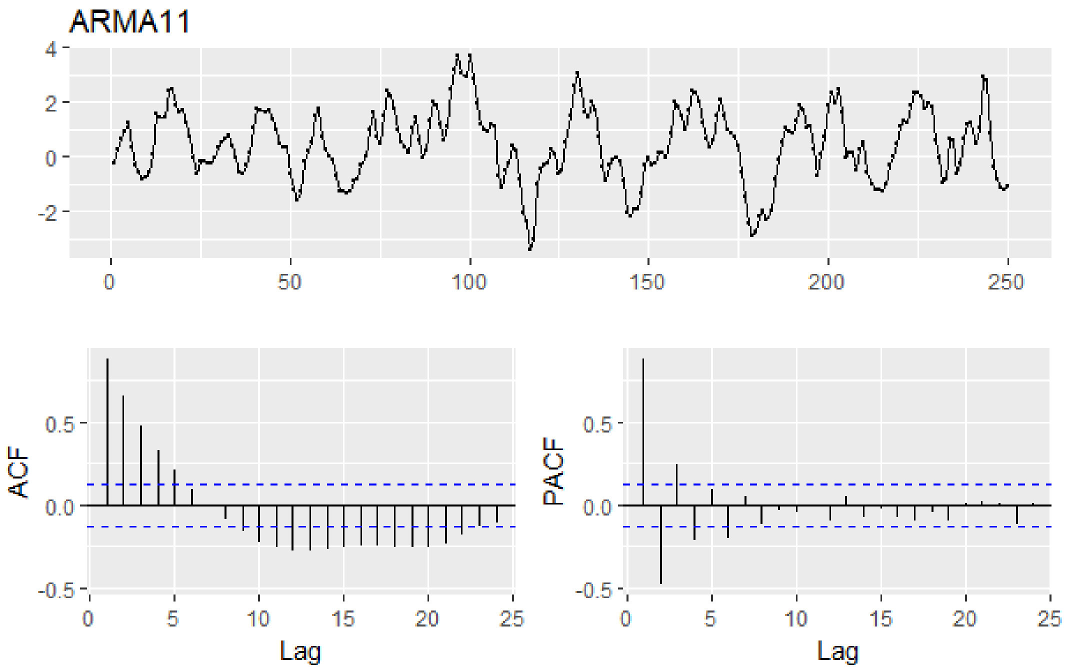


Table entry is the point X^2 with the probability p lying above it. The first column gives the degrees of freedom.

	Probability p					
df	**0.1**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**
1	2.70	3.84	5.02	6.63	7.87	10.83
2	4.60	5.99	7.37	9.21	10.59	13.82
3	6.25	7.81	9.34	11.34	12.83	16.27
4	7.77	9.48	11.14	13.27	14.86	18.47
5	9.23	11.07	12.83	15.08	16.75	20.52
6	10.64	12.59	14.44	16.81	18.54	22.46
7	12.01	14.06	16.01	18.47	20.27	24.32
8	13.36	15.50	17.53	20.09	21.95	26.12
9	14.68	16.91	19.02	21.66	23.58	27.88
10	15.98	18.30	20.48	23.20	25.18	29.59
11	17.27	19.67	21.92	24.72	26.75	31.26
12	18.54	21.02	23.33	26.21	28.30	32.91
13	19.81	22.36	24.73	27.68	29.82	34.53
14	21.06	23.68	26.11	29.14	31.31	36.12
15	22.30	24.99	27.48	30.57	32.80	37.70
16	23.54	26.29	28.84	32.00	34.26	39.25
17	24.76	27.58	30.19	33.40	35.71	40.79
18	25.98	28.86	31.52	34.80	37.15	42.31
19	27.20	30.14	32.85	36.19	38.58	43.82
20	28.41	31.41	34.17	37.56	39.99	45.31
21	29.61	32.67	35.47	38.93	41.40	46.80
22	30.81	33.92	36.78	40.28	42.79	48.27
23	32.00	35.17	38.07	41.63	44.18	49.73
24	33.19	36.41	39.36	42.98	45.55	51.18
25	34.38	37.65	40.64	44.31	46.92	52.62
26	35.56	38.88	41.92	45.64	48.29	54.05
27	36.74	40.11	43.19	46.96	49.64	55.48
28	37.91	41.33	44.46	48.27	50.99	56.89
29	39.08	42.55	45.72	49.58	52.33	58.30
30	40.26	43.77	46.98	50.89	53.67	59.70
40	51.81	55.76	59.34	63.69	66.77	73.40
50	63.17	67.50	71.42	76.15	79.49	86.66
60	74.40	79.08	83.30	88.38	91.95	99.61
70	85.53	90.53	95.02	100.43	104.21	112.32
80	96.58	101.88	106.63	112.33	116.32	124.84
90	107.56	113.15	118.14	124.12	128.30	137.21
100	118.50	124.34	129.56	135.81	140.17	149.45

ARMA Model Diagnosis
Residual analysis

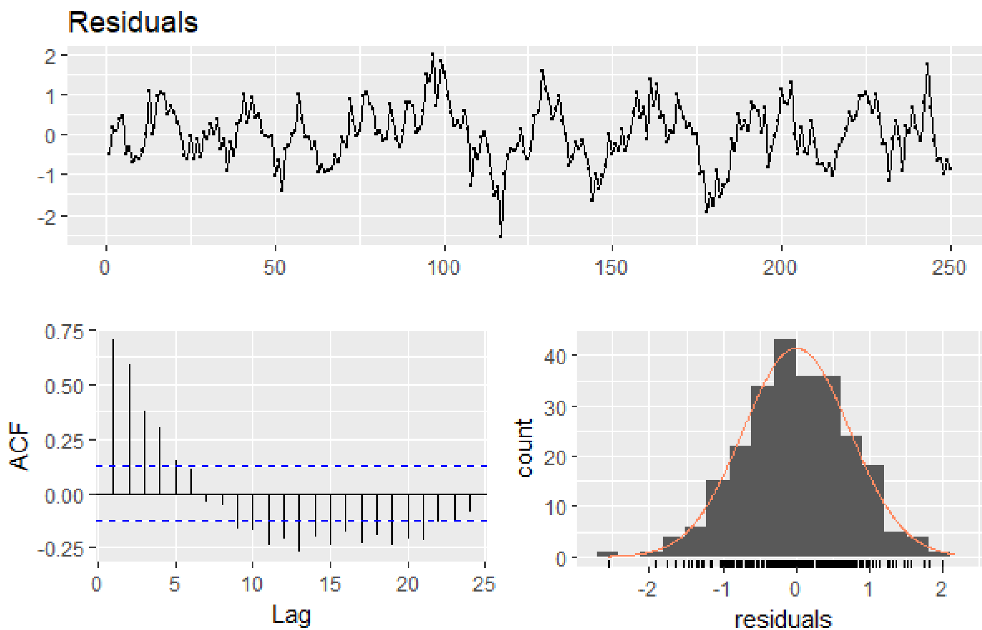
- Analysis of the residuals:
  - Example: $y[t] = 0.8\, y[t-1] - 0.8\, \varepsilon[t-1] + \varepsilon[t]$ with $\varepsilon[t] \sim N(0,0.25)$



ARMA Model Diagnosis
Residual analysis

- Model 1: $\hat{y}[t] = \varphi y[t-1] \quad \Rightarrow \quad \varphi = 0.8899$

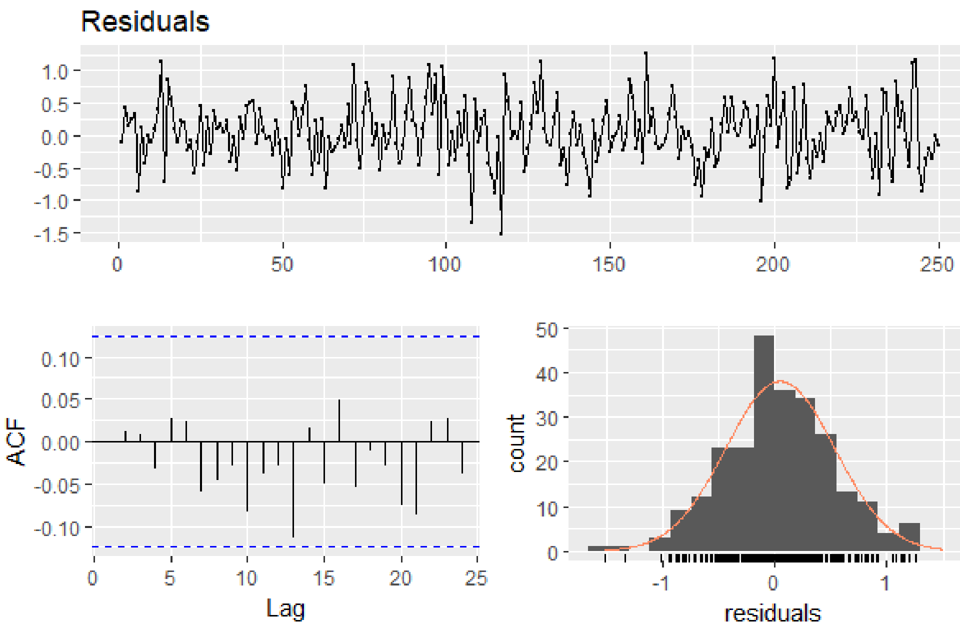
  Box-Ljung test: X-squared = 70.13, df = 20, p-value = 1.734e-07



ARMA Model Diagnosis
Residual analysis

- Model 2: $\hat{y}[t] = \varphi y[t-1] - \theta e[t-1] \Rightarrow \varphi = 0.7854 \;\; \theta = 0.8414$

  Box-Ljung test. X-squared = 12.262, df = 20, p-value = 0.9067



ARMA Model Diagnosis
Level of significance of the coefficients

- t^* statistic: H_0: $\phi_1 = 0$

$$t*_{N-p-q-\delta} = \frac{\hat{\varphi}_1 - (\varphi_1/H_0)}{\hat{\sigma}_{\hat{\varphi}_1}} = \frac{\hat{\varphi}_1}{\hat{\sigma}_{\hat{\varphi}_1}}$$

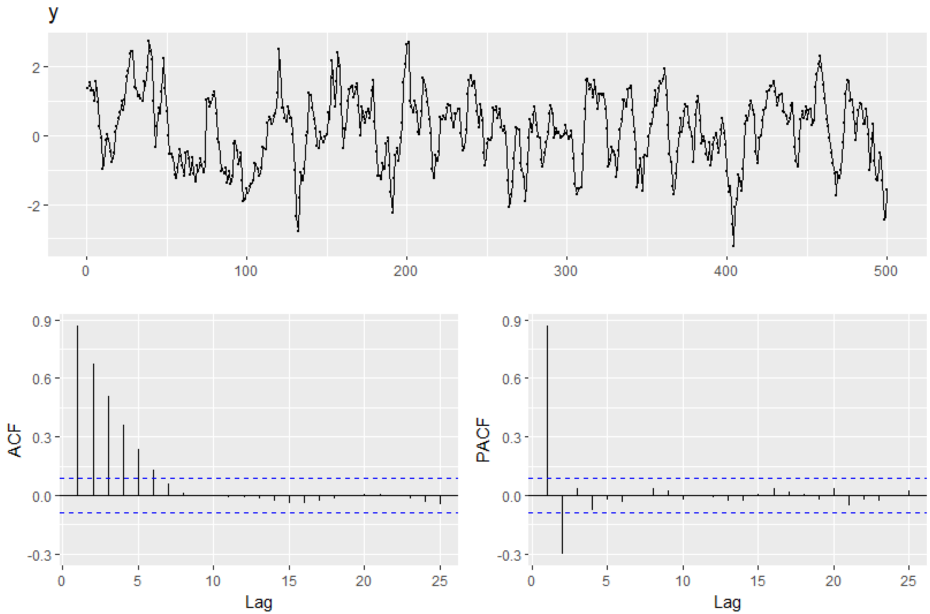
with $\delta=1$ if a constant term is included

```
z test of coefficients:

            Estimate Std. Error z value Pr(>|z|)
ar1         0.742446   0.041857 17.7375   <2e-16 ***
ma1         0.019457   0.064988  0.2994   0.7646
intercept -0.027753   0.056188 -0.4939   0.6214
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

ARMA Model Diagnosis
Example

ARMA(1,1): AR1=0.8, MA1=0.3, s_e=0.25, N=500

```
> y <- arima.sim(n = 500, list(ar = c(0.8), ma = c(0.3)), sd = sqrt(0.25))
> ggtsdisplay(y,lag.max = 25)
```

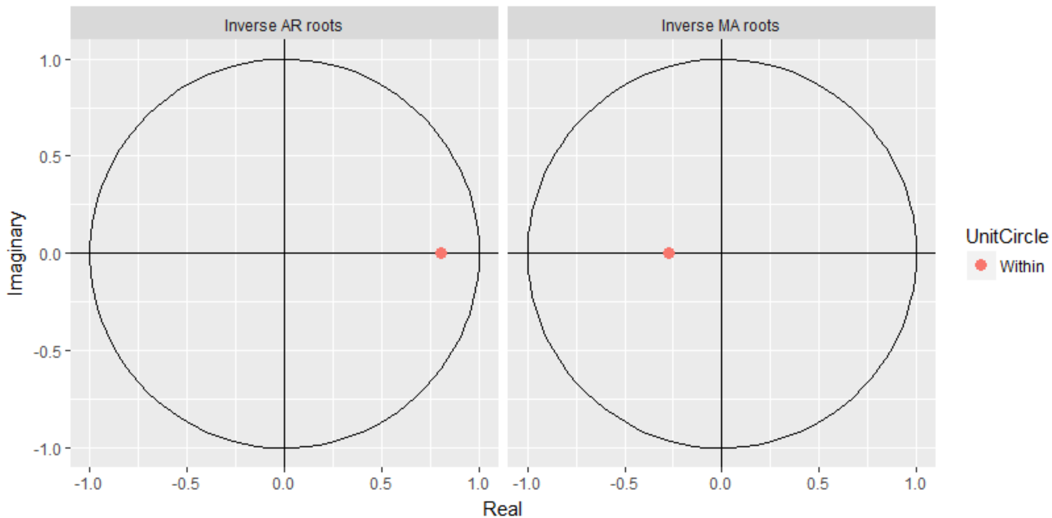


ARMA Model Diagnosis Example

```
> #Fit model
> arima.fit <- Arima(y, order=c(1,0,1), include.constant = TRUE)
> summary(arima.fit)

Series: y
ARIMA(1,0,1) with non-zero mean

Coefficients:
         ar1     ma1    mean
      0.8078  0.2710  0.2557
s.e.  0.0295  0.0482  0.1397

sigma^2 estimated as 0.2286:  log likelihood=-339.74
AIC=687.49   AICc=687.57   BIC=704.34

Training set error measures:
                       ME      RMSE      MAE      MPE     MAPE      MASE        ACF1
Training set 0.0005615644 0.4766461 0.381101 273.1589 406.4277 0.9421676 0.004156234

> coeftest(arima.fit)

z test of coefficients:

          Estimate Std. Error z value  Pr(>|z|)
ar1       0.807751   0.029457 27.4214 < 2.2e-16 ***
ma1       0.271024   0.048205  5.6223 1.885e-08 ***
intercept 0.255653   0.139703  1.8300   0.06725 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

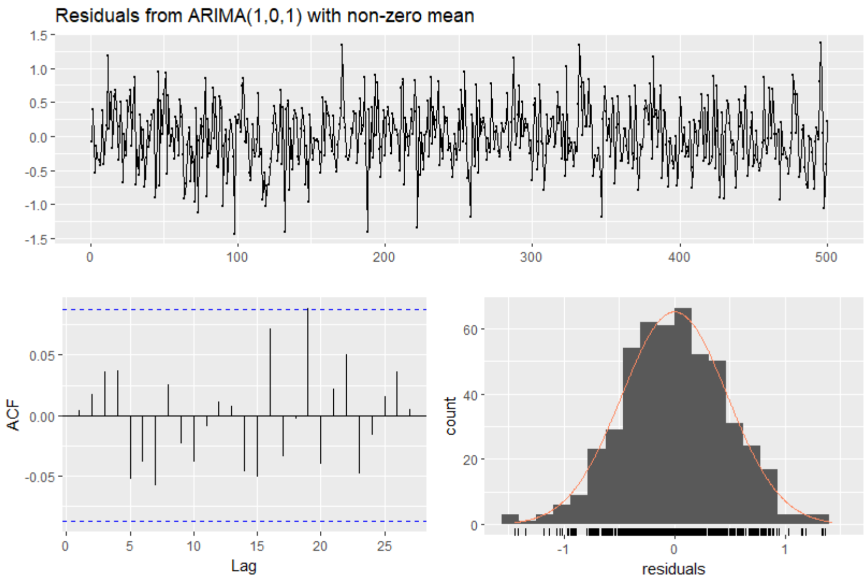
ARMA Model Diagnosis Example

```
> autoplot(arima.fit)
```



ARMA Model Diagnosis Example

```
> checkresiduals(arima.fit)

        Ljung-Box test

data:  Residuals from ARIMA(1,0,1) with non-zero mean
Q* = 6.7621, df = 7, p-value = 0.4541

Model df: 3.   Total lags used: 10
```



ARMA Model Diagnosis
Example

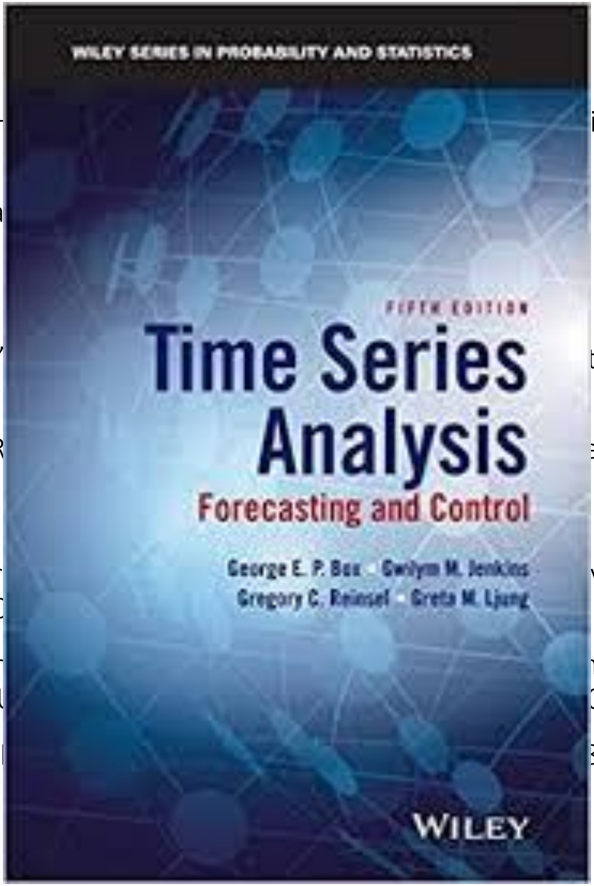
```
> autoplot(y, series="Real")+ forecast::autolayer(arima.fit$fitted, series="Fitted")
```

4

Bibliography

Bibliography

- "Forecasting Principles and Practice". Rob J H ion. Otexts. 2017 (http://otexts.org/fpp3/)
- "Time series analysis. Univariate and Multiva Pearson Addison Wesley. 2006.
- "Análisis de series temporales". Daniel Peña.
- "Time Series Analysis: Forecasting & Control" tice Hall. 1994
- "Princples of Business Forecasting". K. Ord, R ess. 2017
- "Short term Forecasting in Energy. A Guided J. Marín. *Handbook of Power Systems II*. Editor y Iliadis, N.A.. Ed. Springer. Berlin, Germany, 20
- "Analysis of the daily outdoor air temperatur nate change". S. Moreno-Carbonell, E.F. Sánchez-U 0
- "A guide to Delphi". G. Rowe. Foresight: The 3), 11–16. 2007.



Alberto Aguilera 23, E-28015 Madrid - Tel: +34 91 542 2800 - http://www.iit.comillas.edu